The Multivariate Fundamental Theorem of Algebra and Algebraic Geometry

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We derive two consequences of the multivariate fundamental theorem of algebra (MFTA)presented in [1-4]. The first one is the Bezout theorem for n polynomials $g_1, \dots, g_n$ from $k[x_1, \dots, x_n]$. Notably the intersection multiplicities: $I_x(\mathcal{G}),\ (\mathcal{G} = \{g_1, \dots, g_n\})$ as in MFTA, are characterized just by means of partial differential operators given by polynomials from D-invariant linear spaces $D_x(\mathcal{G})$ (see [3]):

$$D_x(\mathcal{G}) := \{p : [D^\alpha p]\,(D) g_i(x) = 0, \quad i = 1, \dots, n, \text{ for all } \alpha \in Z_+^n\},$$

$$I_x(\mathcal{G}) := \dim D_x(\mathcal{G}).$$

Theorem (Bezout). *Let k be an algebraically closed field. Suppose that there is no intersection at "infinity". Then*

$$\sum_x I_x(\mathcal{G}) = \begin{cases} \deg(g_1) \cdots \deg(g_n), & \textit{or} \\ \infty. \end{cases}$$

The case of the intersection at "infinity" can be treated in a standard way.
The second consequence is the following

Theorem. *Let k be an algebraically closed field and $f, f_1, \dots, f_s \in k[x_1, \dots, x_n]$. Then f belongs to the polynomial ideal $< f_1, \dots, f_s >$ if and only if for any $p \in k[x_1, \dots, x_n]$,*

$$[D^\alpha p]\,(D) f_i(x) = 0, \quad i = 1, \dots, s, \ \textit{for all } \alpha \in Z_+^n, \quad \textit{implies}$$

$$[D^\alpha p]\,(D) f(x) = 0, \quad \textit{for all } \alpha \in Z_+^n.$$

Let us mention that one readily gets Nullstellensatz from here. Moreover, the integer m there, i.e., the power of f, can be chosen such that $m \le \deg(f_1) \cdots \deg(f_s) + 1$.

References

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